

Non-Intersecting Arc Matchings on a Square Tile

Marc Dunand

1. The problem, informally

Take a square tile with an even number of points scattered along its four sides. The points are to be joined up in pairs, every point used exactly once. You do not choose the curve joining a pair: it is fixed by where the two points sit, following the three rules below. The only freedom is in *which* point is paired with which.

The pairing has to come out so that no two of the curves touch. The question is whether that can always be arranged, and how.

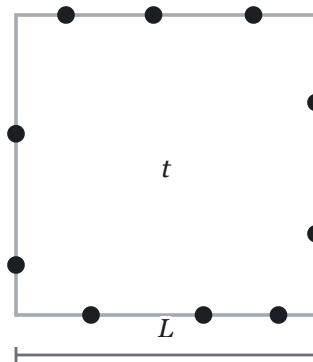


Figure 1. A tile t : a square of side L carrying an even number of points on its four sides. The number on any one side need not be even, only the total.

2. Setup

Let T be a closed square of side L . Its four *sides* are the four closed edge segments; two sides are *adjacent* if they meet at a corner and *opposite* otherwise.

A **tile** t is T together with a finite set P of distinct **points** on its boundary, where $|P| = 2n$ for some integer $n \geq 0$ and no point sits on a corner of T . That second condition is what makes *the side a point lies on* well defined; write $side(p)$ for it.

A **pairing** of t is a perfect matching M of P : a partition of P into n unordered pairs.

3. The arc rules

Each pair $\{a, b\}$ is joined by a curve $\gamma(a, b)$ built from half circles and quarter ellipses. Which one you get depends only on which sides a and b sit on.

Rule 1: both points on the same side

A half circle, centred at the midpoint of ab , with radius half the distance from a to b .

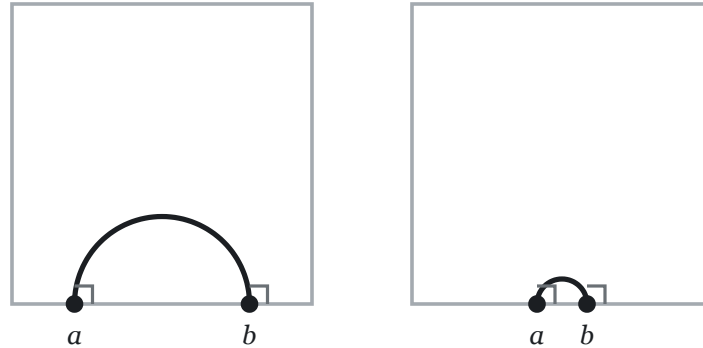


Figure 2. Rule 1. The half circle meets the side at a right angle at both ends, whatever the spacing.

Rule 2: points on two sides that meet at a corner

A quarter ellipse, centred at the corner k those two sides share. Its radii run along the sides, one reaching out to a and the other out to b : that is, $\alpha = |ka|$ and $\beta = |kb|$.

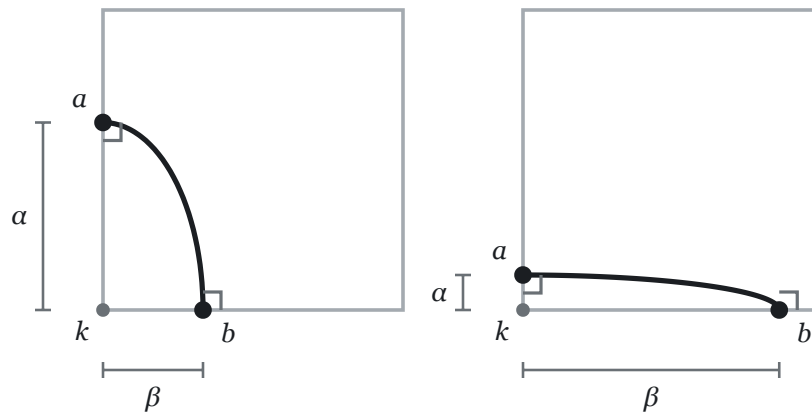


Figure 3. Rule 2. Each radius reaches exactly from the shared corner out to one of the two points, so the quarter ellipse may be close to circular or extremely flat.

Rule 3: points on opposite sides

Let c be the midpoint of the segment ab . The curve is two quarter ellipses meeting at c . The one at a is centred at k_a , the point of a 's side nearest c ; its radii are $|k_a a|$, half the sideways offset between a and b , and $|k_a c|$, which is always half the tile. The one at b is the same with b in place of a .

If a and b sit directly opposite one another the first radius is zero and the curve is the straight segment ab .

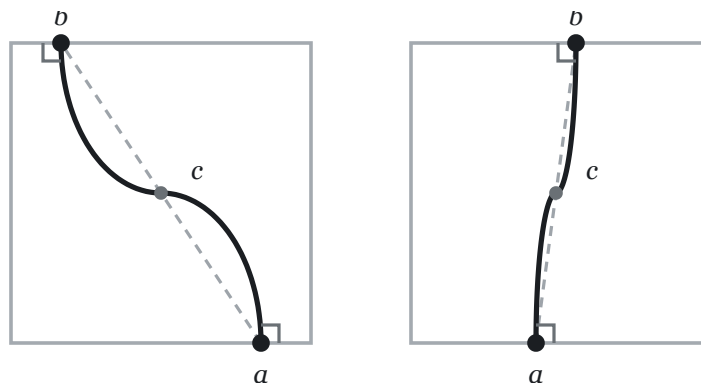


Figure 4. Rule 3. The dashed chord fixes the midpoint c ; each half is then a quarter ellipse from an endpoint to c . The two halves share a tangent there, so the curve is smooth.

Every one of these curves meets the side it lands on at a **right angle**. That is why the rules take the shape they do: curves in two tiles placed side by side then join into a single smooth line across the edge they share, without either tile knowing what the other drew.

They also **stay inside their own tile** without being told to. A half circle reaches a depth of at most half its own side, a quarter ellipse sits in the rectangle of its two radii at the corner, and a Rule 3 curve sits in the rectangle spanned by a and b .

Rule 4: no intersections

The curves must be pairwise disjoint: for distinct pairs $\{a, b\}$ and $\{a', b'\}$ in M ,

$$\gamma(a, b) \cap \gamma(a', b') = \emptyset.$$

Because M is a perfect matching the four endpoints are distinct, so this rules out touching as well as crossing.

4. The task

Give an algorithm which, for every tile t , produces a pairing M of its points satisfying Rules 1 to 4. Equivalently: show that a valid pairing always exists, and construct one.

A solution should handle every tile, not only those in general position, and ideally run in time polynomial in n .

Rule 4 is where the difficulty sits. Join each pair with a straight line instead of its curve, and pairings in which no two of those lines cross are easy to find. Those are not the same thing. A curve is rigid: name its two points and its shape is settled, and it may bulge a long way into the tile on the way between them. So a pairing whose straight lines all miss each other can still draw curves that run through one another.

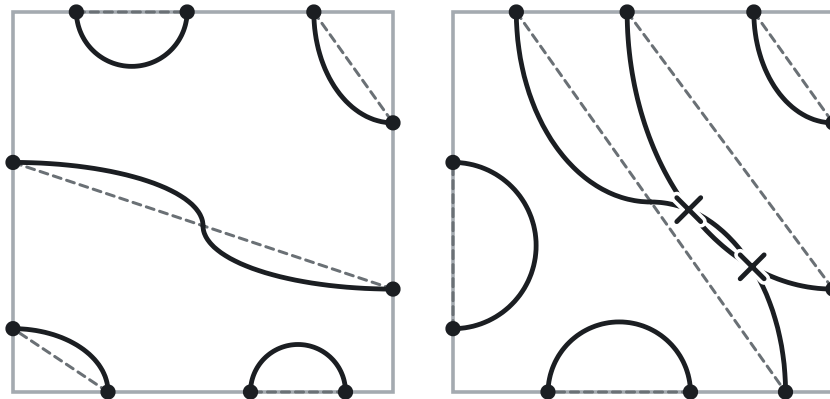


Figure 5. The tile of Figure 1, paired two ways. In both, the dashed links between paired points stay clear of one another. On the left the curves do too, and all four rules hold. On the right they do not, and the intersections are marked.